The Riemann Zeta Function Theory And Applications Aleksandar Ivic


In this thesis we provide a body of knowledge that concerns Riemann zeta-functions and their generalizations in a cohesive manner. In particular, we have studied and mentioned some recent results regarding Hurwitz and Lerch functions, as well as Dirichlet's L-function. We have also investigated some fundamental concepts related to these functions and their universality properties. In addition, we also discuss different formulations and approaches to the proof of the Prime Number Theorem and the Riemann Hypothesis. These two topics constitute the main theme of this thesis. For the Prime Number Theorem, we provide a thorough discussion that compares and contrasts Norbert Wiener's proof with that of Newman's short proof. We have also related them to Hadamard's and de la Vallée Poussin's original proofs written in 1896. As far as the Riemann Hypothesis is concerned, we discuss some recent results related to equivalent formulations of the Riemann Hypothesis as well as the Generalized Riemann Hypothesis. keywords: Riemann zeta function, Hurwitz zeta function, L-functions, Dedekind zeta function, Universality, Prime Number Theorem, Riemann Hypothesis, Generalized Riemann Hypothesis, Analytic Number Theory, Special Functions.

This is an advanced text on the Riemann zeta-function, a continuation of the author's earlier book. It presents the most recent results on mean values, many of which had not yet appeared in print at the time of the writing of the text. An especially detailed discussion is given of the second and the fourth moment, and the latter is studied by the use of spectral theory, one of the most powerful methods used lately in analytic number theory. The book presupposes a reasonable knowledge of zeta-function theory and complex analysis. It will be of great use to the researchers in the field, as well as to all those who wish to get well acquainted with the subject or who have the need for application of zeta-function theory.

In the second part of the book, we start with classical analytical theory of Lerch zeta functions. The book contains both analytic and probabilistic theory of Lerch zeta functions. The text is suitable for a graduate course or seminar or simply as a reference for anyone interested in this extraordinary conjecture.

The Lerch zeta-function is the first monograph on this topic, which is a generalization of the classical Riemann, and Hurwitz zeta-functions. Although analytic results have been presented previously in various monographs on zeta-functions, this is the first book containing both analytic and probabilistic theory of Lerch zeta functions. The book starts with classical analytical theory (Euler gamma functions, functional equation, mean square). The majority of the presented results are new: on approximate functional equations and its applications and on zero distribution (zero-free regions, number of nontrivial zeros etc). Special attention is given to limit theorems in the sense of the weak convergence of probability measures for the Lerch zeta-function. From limit theorems in the space of analytic functions the universality and functional independence is derived. In this respect the book continues the research of the first author presented in the monograph Limit Theorems for the Riemann zeta-function. This book will be useful to researchers and graduate students working in analytic and probabilistic number theory, and can also be used as a textbook for postgraduate students.

Superb high-level study of one of the most influential classics in mathematics examines landmark 1859 publication entitled 'On the Number of Primes Less Than a Given Magnitude.' and traces developments in theory inspired by it. Topics include Riemann's main formula, the prime number theorem, the Riemann-Siegel formula, large-scale computations, Fourier analysis, and other related topics. English translation of Riemann's original document appears in the Appendix.

The subject of this book is probabilistic number theory. In a wide sense probabilistic number theory is part of the analytic number theory, where the methods and ideas of probability theory are used to study the distribution of values of arithmetical objects. This is usually complicated, as it is difficult to say anything about their concrete values. This is why the following problem is usually investigated: given some set, how often do values of an arithmetical object get into this set? It turns out that this frequency follows strict mathematical laws. Here we discover an analogy with quantum mechanics where it is impossible to describe the chaotic behaviour of one particle, but that large numbers of particles obey statistical laws. The objects of investigation of this book are Dirichlet series, and, as the title shows, the main attention is devoted to the Riemann zeta-function. In studying the distribution of values of Dirichlet series the weak convergence of probability measures on different spaces (one of the principle asymptotic probability theory methods) is used. The application of this method was launched by H. Bohr in the third decade of this century and it was implemented in his works together with B. Jessen. Further development of this idea was made in the papers of B. Jessen and A. Wintner, V. Borodzinski and B. Jessen.

Exploring the Riemann Zeta Function: 190 years from Riemann's Birth presents a collection of chapters contributed by eminent experts devoted to the Riemann Zeta Function, its generalizations, and their various applications to several scientific disciplines, including Analytic Number Theory, Harmonic Analysis, Complex Analysis, Probability Theory, and related subjects. The book focuses on both old and new results towards the solution of long-standing problems as well as it features some key historical remarks. The purpose of this volume is to present in a unified way broad and deep areas of research in a self-contained manner. It will be particularly useful for graduate courses and seminars as well as it will make an excellent reference tool for graduate students and researchers in Mathematics, Mathematical Physics, Engineering and Cryptography.
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This volume provides a systematic survey of almost all the equivalent assertions to the functional equations - zeta symmetry - which zeta-functions satisfy, thus streamlining previously published results on zeta-functions. The equivalent relations are given in the form of modular relations in Fox H-function series, which at present include all that have been considered as candidates for ingredients of a series. The results are presented in a clear and simple manner for readers to readily apply without much knowledge of zeta-functions. This volume aims to keep a record of the 150-year-old heritage starting from Riemann on zeta-functions, which are ubiquitous in all mathematical sciences, wherever there is a notion of the norm. It provides almost all possible equivalent relations to the zeta-functions without requiring a reader's deep knowledge on their definitions. This can be an ideal reference book for those studying zeta-functions.

This book provides a lucid exposition of the connections between non-commutative geometry and the famous Riemann Hypothesis, focusing on the theory of one-dimensional varieties over a finite field. The reader will encounter many important aspects of the theory, such as Bombieri's proof of the Riemann Hypothesis for function fields, along with an explanation of the connections with Nevanlinna theory and non-commutative geometry. The connection with non-commutative geometry is given special attention, with a complete determination of the Weil terms in the explicit formula for the point counting function as a trace of a shift operator on the additive space, and a discussion of how to obtain the explicit formula from the action of the ideal class group on the space of adele classes. The exposition is accessible at the graduate level and above, and provides a wealth of motivation for further research in this area.

An introduction to the analytic techniques used in the investigation of zeta functions through the example of the Riemann zeta function. It emphasizes central ideas of broad application, avoiding technical results and the customary function-theoretic approach to the subject.

This book provides a brief overview of the Riemann Zeta function, Riemann Hypothesis and the Hilbert-Polya spectral operator approach to proving RH. Also included in this book is a new discovery that describes a correlation between the Riemann Xi function and gravity rotational curves. Surprisingly their is a mathematical correlation between the complex system of the Riemann Xi function and the large scale distribution of galaxies and rotational curves. Also included in this book are new discoveries on the Prime Number theorem, Riemann Zeta function and other new science and math discoveries.

Zeta functions have been a powerful tool in mathematics over the last two centuries. This book considers a new class of non-commutative zeta functions which encode the structure of the subgroup lattice in infinite groups. The book explores the analytic behaviour of these functions together with an investigation of functional equations. Many important examples of zeta functions are calculated and recorded providing an important data base of explicit examples and methods for calculation.

The aim of the series is to present new and important developments in pure and applied mathematics. Well established in the community over two decades, it offers a large library of mathematics including several important classics. The volumes supply thorough and detailed expositions of the methods and ideas essential to the topics in question. In addition, they convey their relationships to other parts of mathematics. The series is addressed to advanced readers wishing to thoroughly study the topic. Editorial Board Lev Birbrair, Universidade Federal do Ceará, Fortaleza, Brazil Victor P. Maslov, Russian Academy of Sciences, Moscow, Russia Walter D. Neumann, Columbia University, New York, USA Markus J. Pflaum, University of Colorado, Boulder, USA Dierk Schleicher, Jacobs University, Bremen, Germany

Like a hunter who sees 'a bit of blood' on the trail, that's how Princeton mathematician Peter Sarnak describes the feeling of chasing an idea that seems to have a chance of success. If this is so, then the jungle of abstractions that is mathematics is full of frenzied hunters these days. They are out stalking big game: the resolution of 'The Riemann Hypothesis', seems to be in their sights. The Riemann Hypothesis is about the prime numbers, the fundamental numerical elements. Stated in 1859 by Professor Bernhard Riemann, it proposes a simple law which Riemann believed a 'very likely' explanation for the way in which the primes are distributed among the whole numbers, indivisible stars scattered without end throughout a boundless numerical universe. Just eight years later, at the tender age of thirty-nine Riemann would be dead from tuberculosis, cheated of the opportunity to settle his conjecture. For over a century, the Riemann Hypothesis has stumped the greatest of mathematical minds, but these days frustration has begun to give way to excitement. This unassuming comment is revealing astounding connections among nuclear physics, chaos and number theory, creating a frenzy of intellectual excitement amplified by the recent promise of a one million dollar bounty. The story of the quest to settle the Riemann Hypothesis is one of scientific exploration. It is peopled with solitary hermits and gregarious cheerleaders, cool calculators and wild-eyed visionaries, Nobel Prize-winners and Fields Medalists. To delve into the Riemann Hypothesis is to gain a window into the world of modern mathematics and the nature of mathematics research. Stalking the Riemann Hypothesis will open wide this window so that all may gaze through it in amazement.

This text covers exponential integrals and sums, 4th power moment, zero-free region, mean value estimates over short intervals, higher power moments, omega results, zeros on the critical line, zero-density estimates, and more. 1985 edition.

This is an advanced text on the Riemann zeta-function, a continuation of the author's earlier book. It presents the most recent results on mean values, many of which had not yet appeared in print at the time of the writing of the text. An especially detailed discussion is given of the second and the fourth moment, and the latter is studied by the use of spectral theory, one of the most powerful tools in the subject of the book. The book presupposes a reasonable knowledge of zeta-function theory and complex analysis. It will be of great use to the researchers in the field, an to all those who wish to get well acquainted with the subject or who have the need for application of zeta-function theory.

This is the first introductory book on multiple zeta functions and multiple polylogarithms which are the generalizations of the Riemann zeta function and the classical polylogarithms, respectively, to the multiple variable setting. It contains all the basic concepts and the important properties of these functions and their special values. This book is aimed at graduate students, mathematicians and physicists who are interested in this current active area of research. The book will provide a detailed and comprehensive introduction to these objects, their fascinating properties and interesting relations to other mathematical subjects, and various generalizations such as their q-analogs and their finite versions (by taking partial sums modulo suitable prime powers). Historical notes and exercises are provided at the end of each chapter. Contents: Multiple Zeta Functions and Multiple Polylogarithms (MPLs) Multiple Zeta Values (MZVs) Str评ieland Associator and Single-Valued MZVs Multiplie Zeta Values Identities Symmetrized Multiple Zeta Values (SMZVs) Multiple Harmonic Sums (MHSs) and Alternating Version Finite Multiple Zeta Values and Finite Euler Sums Ance of Multiple Harmonic (Star) Sums Readership: Advanced undergraduates and graduate students in mathematics, mathematicians interested in multiple zeta values.

Key Features: For the first time, a detailed explanation of the theory of multiple zeta values is given in book form along with numerous illustrations in explicit examples. The book provides for the first time a comprehensive introduction to multiple polylogarithms and their special values at roots of unity, from the basic definitions to the more advanced topics in current active research. The book contains a few quite intriguing results relating the special values of multiple zeta functions and multiple polylogarithms to other branches of mathematics and physics, such as knot theory and the theory of motives. Many exercises contain supplementary material which deepens the reader's understanding of the main text.
Zeta-function regularization is a powerful tool in perturbation theory, and this book is a comprehensive guide for practitioners in this field. Everything is explained in detail, in particular the mathematical difficulties and tricky points, and several applications are given to show how the procedure works in practice, for example in the Casimir effect, gravity and string theory, high-temperature phase transition, topological symmetry breaking, and non-commutative spacetime. The formulae, some of which are new, can be directly applied in creating physically meaningful, accurate numerical calculations. The book acts both as a basic introduction and a collection of exercises for those who want to apply this regularization procedure in practice. Thoroughly revised, updated and expanded, this new edition includes novel, explicit formulae on the general quadrature, the Chowla-Selberg series case, an interplay with the Hadamard calculus, and also features a fresh chapter on recent cosmological applications, including the calculation of the vacuum energy fluctuations at large scale in brane-world and other models.

The Riemann zeta function was introduced by L. Euler (1737) in connection with questions about the distribution of prime numbers. Later, B. Riemann (1859) derived deeper results about the prime numbers by considering the zeta function in the complex variable. The famous Riemann Hypothesis, asserting that all of the non-trivial zeros of zeta are on a critical line in the complex plane, is one of the most important unsolved problems in modern mathematics. The present book consists of two parts. The first part covers classical material about the zeros of the zeta function with applications to the distribution of prime numbers, including those made by Riemann himself, F. Carlson, and Hardy-Littlewood. The second part gives a complete presentation of Levinson's method for zeros on the critical line, which allows one to prove, in particular, that more than one-third of non-trivial zeros of zeta are on the critical line. This approach and some results concerning integrals of Dirichlet polynomials are new. There are also technical lemmas which can be useful in a broader context.

In this text, the famous zeros of the Riemann zeta function and its generalizations (L-functions, Dedekind and Selberg zeta functions) are analyzed through several zeta functions built over those zeros.

Studying the relationship between the geometry, arithmetic and spectra of fractals has been a subject of significant interest in contemporary mathematics. This book contributes to the literature on the subject in several different and new ways. In particular, the authors provide a rigorous and detailed study of the spectral operator, a map that sends the geometry of fractal strings onto their spectrum. To that effect, they use and develop methods from fractal geometry, functional analysis, complex operator theory, partial differential equations, analytic number theory and mathematical physics. Originally, M. L. Lapidus and M. van Frankenhuysen "heuristically" introduced the spectral operator in their development of the theory of fractal strings and their complex dimensions, specifically in their reinterpretation of the earlier work of M. L. Lapidus and H. Maier on inverse spectral problems for fractal strings and the Riemann hypothesis. One of the main themes of the book is to provide a rigorous framework within which the corresponding question "Can one hear the shape of a fractal string?" or, equivalently, "Can one obtain information about the geometry of a fractal string, given its spectrum?" can be further reformulated in terms of the invertibility or the quasi-invertibility of the spectral operator. The infinitesimal shift of the real line is first precisely defined as a differentiation operator on a family of suitably weighted Hilbert spaces of functions on the real line and indexed by a dimensional parameter c. Then, the spectral operator is defined via the functional calculus as a function of the infinitesimal shift. In this manner, it is viewed as a natural "quantum" analog of the Riemann zeta function. More precisely, within this framework, the spectral operator is defined as the composite map of the Riemann zeta function with the infinitesimal shift, viewed as an unbounded normal operator acting on the above Hilbert space. It is shown that the quasi-invertibility of the spectral operator is intimately connected to the existence of critical zeros of the Riemann zeta function, leading to a new spectral and operator-theoretic reformulation of the Riemann hypothesis. Accordingly, the spectral operator is quasi-invertible for all values of the dimensional parameter c in the critical interval (0,1) (other than in the midfractal case when c =1/2) if and only if the Riemann hypothesis (RH) is true. A related, but seemingly quite different, reformulation of RH, due to the second author and referred to as an "asymmetric criterion for RH," is also discussed in some detail: namely, the spectral operator is invertible for all values of c in the left-critical interval (0,1/2) if and only if RH is true. These spectral reformulations of RH also led to the discovery of several "mathematical phase transitions" in this context, for the shape of the spectrum, the invertibility, the boundedness or the unboundedness of the spectral operator, and occurring either in the midfractal case or in the most fractal case when the underlying fractal dimension is equal to 1/2 or 1, respectively. In particular, the midfractal dimension c =1/2 is playing the role of a critical parameter in quantum statistical physics and the theory of phase transitions and critical phenomena. Furthermore, the authors provide a "quantum analog" of Voronin's classical theorem about the universality of the Riemann zeta function. Moreover, they obtain and study quantized counterparts of the Dirichlet series and of the Euler product for the Riemann zeta function, which are shown to converge (in a suitable sense) even inside the critical strip. For pedagogical reasons, most of the book is devoted to the study of the quantized Riemann zeta function. However, the results obtained in this monograph are expected to lead to a quantization of most classical arithmetic zeta functions, hence, further "naturally quantizing" various aspects of analytic number theory and arithmetic geometry. The book should be accessible to experts and non-experts alike, including mathematics and physics graduate students and postdoctoral researchers, interested in fractal geometry, number theory, operator theory and functional analysis, differential equations, complex analysis, spectral theory, as well as mathematical and theoretical physics. Whenever necessary, suitable background about the different subjects involved is provided and the new work is placed in its proper historical context. Several appendices supplementing the main text are also included.

The Riemann zeta-function is our most important tool in the study of prime numbers, and yet the famous "Riemann hypothesis" at its core remains unsolved. This book studies the theory from every angle and includes new material on recent work.

The authors demonstrate that the Dirichlet series representation of the Riemann zeta function converges geometrically at the roots in the critical strip. The Dirichlet series parts of the Riemann zeta function diverge everywhere in the critical strip. It has therefore been assumed for at least 150 years that the Dirichlet series representation of the zeta function is useless for characterization of the non-trivial roots. The author shows that this assumption is completely wrong. Reduced, or simplified, asymptotic expansions of the terms of the zeta function series parts are equated algebraically with reduced asymptotic expansions for the terms of the Dirichlet zeta-function. For example, the canonical factorization theorem involves an analogue of the Euler constant. Finally the logarithmic derivative of the Selberg zeta-function. Previous knowledge of the Selberg trace formula is not assumed. The theory is developed for arbitrary real weights and for arbitrary multiplier systems permitting an approach to known results on classical and automorphic forms without the Riemann-Roch theorem. The author's discussion of the Selberg trace formula stresses the analogy with the Riemann zeta-function. For example, the canonical factorization theorem involves an analogue of the Euler constant. Finally the logarithmic derivative of the Selberg zeta-function. Previous knowledge of the Selberg trace formula is not assumed. The theory is developed for arbitrary real weights and for arbitrary multiplier systems permitting an approach to known results on classical and automorphic forms without the Riemann-Roch theorem. The author's discussion of the Selberg trace formula stresses the analogy with the Riemann zeta-function. For example, the canonical factorization theorem involves an analogue of the Euler constant. Finally the logarithmic derivative of the Selberg zeta-function.

This book provides both classical and new results in Riemann Zeta-Function theory, one of the most important problems in analytic number theory. These results have application in solving problems in multiplicative number theory, such as power moments, the zero-free region, and the zero density estimates. The book also furnishes annotated proofs, end-of-chapter notes, historical discussions and references.

The Notes give a direct approach to the Selberg zeta-function for cofinite discrete subgroups of SL (2, R) acting on the upper half-plane. The basic idea is to compute the trace of the iterated resolvent kernel of the hyperbolic Laplacian in order to arrive at the logarithmic derivative of the Selberg trace formula. Previous knowledge of the Selberg trace formula is not assumed. The theory is developed for arbitrary real weights and for arbitrary multiplier systems permitting an approach to known results on classical and automorphic forms without the Riemann-Roch theorem. The author's discussion of the Selberg trace formula stresses the analogy with the Riemann zeta-function. For example, the canonical factorization theorem involves an analogue of the Euler constant. Finally the general Selberg trace formula is deduced easily from the properties of the Selberg zeta-function: this is similar to the procedure in analytic number theory where the explicit formulae are deduced from the properties of the Riemann zeta-function. Apart from the basic spectral theory of the Laplacian for cofinite groups the book is self-contained and will be useful as a quick approach to the Selberg zeta-function and the Selberg trace formula.

This volume contains the proceedings of the conference "Casimir Force, Casimir Operators and the Riemann Hypothesis | Mathematics for Innovation in Industry and Science" held in November 2009 in Fukuoka (Japan). The conference focused on the following topics: Casimir operators in harmonic analysis and representation theory Number theory, in particular zeta functions and cryptography Casimir force in physics and its relation with nano-science Mathematical biology Importance of mathematics for innovation in industry.